String-localized quantum fields from Wigner representations

Jens Mund*

Instituto de Física, Universidade de São Paulo,

CP 66 318, 05315 - 970 São Paulo, SP, Brazil

Bert Schroer†

CBPF, Rua Dr. Xavier Sigaud 150,

22290-180 Rio de Janeiro, Brazil, and

Institut für Theoretische Physik, FU-Berlin,

Arnimallee 14, D-14195 Berlin, Germany

Jakob Yngvason‡

Institut für Theoretische Physik, Universität Wien,

Boltzmanngasse 5, A-1090 Vienna, Austria, and

Erwin Schrödinger Institute for Mathematical Physics,

Boltzmanngasse 9, A-1090 Vienna, Austria

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Abstract

In contrast to the usual representations of of the Poincaré group of finite spin or helicity the Wigner representations of mass zero and infinite spin are known to be incompatible with pointlike localized quantum fields. We present here a construction of quantum fields associated with these representations that are localized in semi-infinite, space-like strings. The construction is based on concepts outside the realm of Lagrangian quantization with the potential for further applications.

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It is well-known that free fields for particles of finite spin (or helicity in case of $m = 0$) can be constructed in two ways, either by (canonical or functional integral) Lagrangian quantization, or within the setting of Wigner's particle classification [1] based on positive energy representations of the universal covering of the Poincaré group [2]. There is, however, a family of representations where the standard field-theoretical procedures fail. These representations correspond to particles of zero mass and infinite spin and can be regarded as limiting cases of representations of mass $m > 0$ and spin $s < \infty$ as $m \to 0$ and $s \to \infty$ with the Pauli-Lubanski parameter $m^2 s(s+1) = \kappa^2$ fixed and nonzero. In the Wigner classifications they are associated with faithful representations of the noncompact stabilizer group ("little group") of a light-like vector. In this case no Lagrangian description is known; in fact there exists a No-Go theorem [3] stating that these representations are incompatible with pointlike localized fields fulfilling the general principles of quantum field theory [4]. Special examples that indicate the difficulties to make these representations compatible with the structure of local fields can also be found in [5, 6].

In this Letter we report on the construction of *string-localized* fields for these representations; the string turns out to be a semi-infinite space-like line characterized by an initial point x in Minkowski space and a space-like direction e from the unit space-like hyperboloid (a point in a de Sitter space). In this paper "localization" is always understood in terms of the vanishing or nonvanishing of commutators of field operators, and string-localization means that the commutator of two field operator vanishes if the corresponding strings are space-like separated but in general not if this holds only for the end points. The existence of string-localized objects as the best possible (with the tightest localization) for these representations is suggested by recent general results on localization in space-like cones that apply to all positive energy representations of the Poincaré group [7]. Our string-localized fields transform in a simple way under the Poincaré group and their internal degrees of freedom consist in the infinite helicity tower of a faithful representation of the Euclidean stabilizer group $E(d-2)$ in space-time dimension $d \geq 4$. For $d = 3$ the representation is one-dimensional but leads also to string-localized fields. For concreteness sake we consider here the case $d = 4$ and integer helicities. Our findings solve an old problem that has attracted the attention of physicists of several generations [3, 5, 6, 8], namely to incorporate these representations into quantum field theory in a way compatible with causality.

New concepts, outside the realm of Lagrangean quantization, have been essential for our

construction. We regard our work as an argument in favour of the strength and relevance to QFT of these concepts, which have the potential for further applications as pointed out below.

An interesting feature of our construction is a subtle interplay between the pointlike localization of the end point of the string in d-dimensional Minkowski space and the directional localization in a $(d-1)$-dimensional de Sitter space in the sense of [9]. We note that in his search for a classical local equation for the zero mass infinite spin representations Wigner [8] proposed a description in which the Poincaré group also acts on a space-like vector besides the points in Minkowski space. The wave equations of [8], however, are inconsistent with string-localization in the sense considered here.

The infinite spin Wigner representations are not the only irreducible representations leading to string localization; massive representations in $d = 1 + 2$ with spin not equal to an integer or half-integer (anyons) can only be string localized. In that case the string localization results from the richer covering structure of the $d = 1 + 2$ Poincaré group which also leads to braid group statistics which requires the presence of vacuum polarization even in the absence of a genuine interaction (absence of real particle creation) [10]. The anyonic string is a special case of the string-like localized objects envisaged in [11].

In this context it is worth pointing out that there is a significant difference between string localization in our sense and localization in string field theory. The lightfront quantization of the free bosonic Nambu-Goto string leads according to the analysis in [12] to pointlike localization in the sense that the commutator vanishes for space-like separation of the centers of mass of two string configurations, irrespective of an overlap of their internal coordinates. For interacting string field theory there are no rigorous results of this kind, but perturbative calculations [13, 14] seem to indicate that if such a theory is meaningful at all (which is by no means clear) the string fields can be expected to be totally delocalized. On the available evidence it seems in any case fair to say that the strings of string field theory are not string-localized in the sense of the present paper.

Our construction of string-localized fields is based on Tomita-Takeski modular theory (see [15] for a survey of its applications to quantum field theory) in the context of modular localization for Poincaré covariant positive energy representations [7, 16, 17, 18, 19]. A full treatment in the modular setting will be given in [20]. Here we only describe the main result and give an argument which (in the present condensed version) is less systematic and

rigorous but has the advantage of being more accessible to readers with a standard field-theoretic background. For convenience of the reader we include some basic definitions and facts about modular localization in an appendix.

We start our construction by recalling the definition of the irreducible zero mass, infinite spin representations of the proper, orthochronous Poincaré group $\mathcal{P}_+^\uparrow$. They are defined by inducing unitary representations of the stabilizer group of a fixed light-like vector to the whole of $\mathcal{P}_+^\uparrow$. The stabilizer group is in our case isomorphic to the two-dimensional Euclidean group $E(2)$, consisting of rotations R_ϑ by an angle $\vartheta \in \mathbb{R} \bmod 2\pi$ and translations by $c \in \mathbb{R}^2$. Let $\mathcal{H}_\kappa$ be the Hilbert space of functions of $k \in \mathbb{R}^2$, square integrable with respect to the measure $d\nu_\kappa = \delta(|k|^2 - \kappa^2)d^2k$. (Hence only the restrictions of the functions to a circle of radius κ matter.) The Pauli-Lubanski parameter κ^2 labels nonequivalent representations of $E(2)$; the representation on $\mathcal{H}_\kappa$ is given by the formula

$$(D_\kappa(c, R_\vartheta)\varphi)(k) = e^{ic\cdot k}\varphi(R_\vartheta^{-1}k). \tag{1}$$

Let $\psi(p)$ be an $\mathcal{H}_\kappa$-valued wave function of $p \in \mathbb{R}^4$, square integrable with respect to the Lorentz invariant measure $d\mu(p) = \theta(p^0)\delta(p\cdot p)d^4p$ on the mantle ∂V^+ of the forward light cone V^+. The unitary Wigner transformation law for such a wave function reads

$$(U(a, \Lambda)\psi)(p) = e^{ip\cdot a}D_\kappa(R(\Lambda, p))\psi(\Lambda^{-1}p) \tag{2}$$

where

$$R(\Lambda, p) = B_p^{-1}\Lambda B_{\Lambda^{-1}p} \in E(2) \tag{3}$$

denotes the Wigner "rotation" (actually a boost combined with a rotation) with B_p an appropriately chosen Lorentz transformation that transforms the standard vector $\bar{p} = (1, 0, 0, 1)$ to a (nonzero) $p \in \partial V^+$.

Our string-localized field operators are defined on the Fock-space over the irreducible representation space with the creation and annihilation operators $a^*(p)(k)$, $a(p)(k)$ for the basis kets $|p, k\rangle$ of the one-particle space, $p \in \partial V^+$, $k \in \mathbb{R}^2$, $|k| = \kappa$. In fact, we define a whole family of fields, depending on a complex parameter α that labels representations of the 3-dimensional de Sitter group as will be explained in the sequel. The field operators have the form

$$\Phi^\alpha(x, e) = \int_{\partial V^+} d\mu(p)\left\{e^{ipx}u^\alpha(p, e)\circ a^*(p) \right.$$
$$\left. + e^{-ipx}\overline{u^{\bar{\alpha}}(p, e)}\circ a(p)\right\} \tag{4}$$

with $\mathcal{H}_\kappa$-valued prefactors $u^\alpha(p,e)$ that are determined by the intertwining property (7) below and certain analyticity requirements for their dependence on e. The circle "$\circ$" between the prefactors $u^\alpha(p,e)$ and the creation and annihilation operators (the dependence on k is suppressed by the notation) stands for integration over $k \in \mathbb{R}^2$ with respect to the measure $d\nu_\kappa(k)$, and the bar denotes complex conjugation. The fields are singular in x and the space-like direction e, i.e., operator valued distributions, and they have the following properties:

- If $x + \mathbb{R}^+ e$ and $x' + \mathbb{R}^+ e'$ are space-like separated [29] then

$$\left[\Phi^\alpha(x,e), \Phi^{\alpha'}(x',e')\right] = 0 \tag{5}$$

  while the commutator is nonzero as a distribution in e, e' if only the the endpoints of the strings, x and x', are space-like separated.

- The transformation law of the field is consistent with this localization:

$$U(a,\Lambda)\Phi^\alpha(x,e)U(a,\Lambda)^{-1} = \Phi^\alpha(\Lambda x + a, \Lambda e). \tag{6}$$

- After smearing with tests functions in x and e, where it is sufficient to let x and e vary in an arbitrary small region, the field operators generate a dense set in Fock space when applied to the vacuum vector $|0\rangle$. (Reeh-Schlieder property [4].)

The second statement (6) is a result (as in the standard finite spin case) of the intertwining properties of u^α, namely u^α and $\overline{u^{\bar\alpha}}$ absorb the Wigner rotation of the creation/annihilation operators (which is contragradient to that of the wave function (2)) and trade it for a transformation of e according to

$$D_\kappa(R(\Lambda,p))u^\alpha(\Lambda^{-1}p,e) = u^\alpha(p,\Lambda e). \tag{7}$$

The localization (5), on the other hand, results from (6), TCP covariance, and analyticity properties of the two point function in $x - x'$ and in e, e'. The third property is proved in a similar way as the Reeh-Schlieder theorem for point-localized fields [4], using also analyticity in e. The field operators for different values of the parameter α all generate the same Fock space and Eq. (5) implies that they are relatively (string) localized to each other. Hence they all belong to the same Borchers class [21].

The intuitive basis of this construction is the idea that one can obtain the relevant representation by a suitable projection from a tensor product representation, where one

factor is a scalar massless Wigner representation of the Poincaré group in $d = 4$ dimensional Minkowski space and the other a representation of the Lorentz group associated with a $d-1 = 3$ dimensional de Sitter space. Without any relation between the tensor factors, one would obtain a factorizing two-point function associated with a commutator that vanishes if both the Minkowski- and de Sitter localizations points are space-like. The action of the Poincaré group in the tensor product space $\mathcal{H} = \mathcal{H}_0 \bar{\otimes} \mathcal{H}_{\mathrm{dS}}$ is $U_{\mathrm{tens}}(a,\Lambda) = U_0(a,\Lambda)\bar{\otimes} U_{\mathrm{dS}}(\Lambda)$, where $U_0(a,\Lambda)$ is the Wigner representation of a massless, scalar particle, and $U_{\mathrm{dS}}(\Lambda)$ is a representation of the homogenous Lorentz-group on functions on $d-1$ dimensional de Sitter space as in [9] of degree α, which is unitary if $\alpha = -\frac{d-2}{2} + i\rho$, $\rho \in \mathbb{R}$ [30]. It turns out that for our purpose all values of α are allowed (except $\alpha = 0, 1, 2, \ldots$ for which $u^\alpha \equiv 0$ for $k \neq 0$ by Eqs. (8) and (9) below), but the unitary case, $\mathrm{Re}\,\alpha = -1$, is perhaps the most natural choice. For unitary U_{dS} the representation $U_{\mathrm{tens}}(a,\Lambda)$ is a direct integral of the continuum of infinite spin Wigner representations corresponding to all real values of the Pauli-Lubanski parameter κ. Projecting out one of these uncountably many irreducible representations weakens the independent localizations in x and e in such a way as to be consistent with the mutual causal dependency of strings. The decomposition of the tensor product representation into its irreducible components is carried out by first bringing it into the Wigner form (i.e., the form of (2)) by means of a unitary transformation $\psi(p) \to U_{\mathrm{dS}}(B_p)\psi(p)$ and then decomposing it according to the spectrum of the Casimir operator of the little group. A definite value of κ is then picked out. The resulting intertwiners are

$$u^\alpha(p,e)(k) = e^{-i\pi\alpha/2} \int d^2 z e^{ik\cdot z} \left(B_p \xi(z) \cdot e\right)^\alpha \tag{8}$$

with

$$\xi(z) = \left(\tfrac{1}{2}\left(|z|^2 + 1\right), z_1, -z_2, \tfrac{1}{2}\left(|z|^2 - 1\right)\right). \tag{9}$$

Here $\xi \in \partial V^+$ is a de Sitter momentum space variable, and $(\xi \cdot e)^\alpha$ (the dot denotes here the Minkowski inner product) is the analog of a plane wave, i.e., as a function of ξ and the exponent α it is the Fourier-Helgason transform of the δ-function at the point e in de Sitter space as explained in [9]. The power t^α is defined with a cut along $\mathbb{R}^-$ and $(-1)^\alpha = \exp(i\pi\alpha)$. Instead of integrating ξ over time-like or space-like cycles Γ as the authors of [9], we chose the light-like cycle $\Gamma_{(1,0,0,1)} = \{\xi \in \partial V^+, (\xi \cdot e) = 1\}$ that leads to the parametrization (9) in terms of points $z \in \mathbb{R}^2$. The integral in (8) is understood in the sense of tempered distributions, but by partial integration one sees that for $k \neq 0$ the result is a continuous

function of $k \in \mathbb{R}^2$ that can be restricted to $|k| = \kappa$.

Since $B_p\xi(z) \in \partial V^+$ has a positive scalar product with any vector in the forward light cone V^+, it follows from (8) that $u^\alpha(p,e)(k)$ can be defined for complex vectors e, provided the imaginary part of e is in V^+. Moreover, $u^\alpha(p,e)(k)$ is analytic in e in this domain.

The nontrivial coupling between initial points and directions arises from the presence of the p-dependent boost B_p and of the 2D plane wave factor $e^{ik\cdot z}$ which produces the variable k on which the Lorentz group acts through the Wigner "rotation" $D_\kappa(R(\Lambda,p))$, c.f. (1) and (3). This action, consisting of a two-dimensional translation c and a rotation R_ϑ both depending on Λ and p (i.e., $R(\Lambda,p) = (c, R_\vartheta)$), can be pulled through to the z in $\xi(z)$ as follows:

$$\begin{aligned}
&D_\kappa(R(\Lambda,p))u^\alpha(\Lambda^{-1}p,e)(k) = e^{ic\cdot k}u^\alpha(\Lambda^{-1}p,e)(R_\vartheta^{-1}k) \\
&= e^{-i\pi\alpha/2}\int d^2z e^{ik\cdot z}\left(B_{\Lambda^{-1}p}R(\Lambda,p)^{-1}\xi\left(z\right)\cdot e\right)^\alpha \\
&= e^{-i\pi\alpha/2}\int d^2z e^{ik\cdot z}\left(\Lambda^{-1}B_p\xi\left(z\right)\cdot e\right)^\alpha = u^\alpha(p,\Lambda e)(k),
\end{aligned} \tag{10}$$

verifying (7). Here we have in the second line used the relation $\xi(R_\vartheta z + c) = R(\Lambda,p)\xi(z)$ that follows directly from the above formula (9) for $\xi(z)$. The passing to the third line uses the formula (3) for Wigner rotation $R(\Lambda,p)$. Besides the representation of $\mathcal{P}_+^\uparrow$ an antiunitary TCP transformation is defined by $|p,k\rangle \rightarrow |p,-k\rangle$, which means that $u^\alpha(p,e)(k) \rightarrow \overline{u^\alpha(p,e)(-k)} = u^{\bar\alpha}(p,-e)(k)$. This sets the stage for the application of the modular localization [7] of one-particle states that can be shown to imply the desired string commutation relation. We shall not discuss this approach here but pass directly to the commutator via the two-point function

$$\begin{aligned}
\mathcal{W}^{\alpha\alpha'}(x-x';e,e') &= \left\langle 0|\Phi^\alpha(x,e)\Phi^{\alpha'}(x',e')|0\right\rangle \\
&= \int_{\partial V^+} d\mu(p)e^{-ip\cdot(x-x')}M^{\alpha\alpha'}(p;e,e'),\\
M^{\alpha\alpha'}(p;e,e') &= \overline{u^{\bar\alpha}(p,e)} \circ u^{\alpha'}(p,e'),
\end{aligned} \tag{11}$$

where $\circ$ again denotes integration over k on the circle $|k| = \kappa$. In contradistinction to pointlike localized fields, where $M^{\alpha\alpha'}$ is a polynomial in p, we cannot express this two-point function in terms of known functions but we can read off its covariance properties from Eq.

(7) and the TCP symmetry in the one-particle space:

$$M^{\alpha\alpha'}(p;\Lambda e,\Lambda e') = M^{\alpha\alpha'}(\Lambda^{-1}p;e,e') \tag{12}$$

$$M^{\alpha\alpha'}(p;-e,-e') = M^{\alpha'\alpha}(p;e',e). \tag{13}$$

Since the measure $d\mu(p)$ has support on ∂V^+ the two-point function $\mathcal{W}^{\alpha\alpha'}(x-x';e,e')$ is an analytic function of $x-x'$ in the complex domain $\mathbb{R}^4 - iV^+$. Moreover, by the analyticity of u^α in e, $\mathcal{W}^{\alpha\alpha'}$ is analytic for complex e' with $e'\cdot e' = -1$ and imaginary part in V^+. Likewise, it is antianalytic for complex e in the same domain.

If two strings, $x+\mathbb{R}^+e$ and $x'+\mathbb{R}^+e'$ are space-like separated (cf. footnote [22]), there is a space-like wedge W with causal complement W' such that $x+\mathbb{R}^+e \in W$ and $x'+\mathbb{R}^+e' \in W'$. By translational invariance of the two-point function it can be assumed that the edge of W (and hence also of W') contains the origin; then $x,e\in W$ and $x',e'\in W'$. The covariance law (12) and the TCP symmetry (13) imply the following "exchange formula":

$$\mathcal{W}^{\alpha'\alpha}(x'-j\Lambda(-t)x;e',j\Lambda(-t)e) = \mathcal{W}^{\alpha\alpha'}(x-j\Lambda(t)x';e,j\Lambda(t)e'). \tag{14}$$

Here j is the reflection across the edge of the wedge W which transforms W into W' and V^+ into $-V^+$, and $\Lambda(t)$ is the one-parameter group of Lorentz boosts that leave W invariant. Note that j and $\Lambda(t)$ commute. The matrix valued function $\Lambda(t)$ is entire analytic in the boost parameter t. Moreover, for t in the strip $\mathbb{R}+i(0,\pi)$ the imaginary parts of $j\Lambda(-t)x$, $j\Lambda(-t)e$, $j\Lambda(t)x'$ and $j\Lambda(t)e'$ all lie in V^+. Eq. (14) extends from the boundary at $\operatorname{Im} t = 0$ to the whole strip by the analyticity of the two point function and the Schwarz reflection principle. The boundary values for $\operatorname{Im} t = i\pi$ are therefore also identical for both sides. Since $j\Lambda(\pm i\pi)$ is the identity matrix, this leads to the desired stringlike commutativity in the form $\mathcal{W}^{\alpha'\alpha}(x'-x;e',e) = \mathcal{W}^{\alpha\alpha'}(x-x';e,e')$ if $x+\mathbb{R}^+e$ and $x'+\mathbb{R}^+e'$ are space-like separated.

The structure of the two-point function also permits the definition of a KMS (thermal equilibrium) state at inverse temperature β, replacing $M^{\alpha\alpha'}(p;e,e')$ by

$$M_\beta^{\alpha\alpha'}(p;e,e') = \left(1-e^{-\beta p^0}\right)^{-1}\left[\theta(p^0)M^{\alpha\alpha'}(p;e,e')\ -\theta(-p^0)M^{\alpha'\alpha}(-p;e',e)\right] \tag{15}$$

with θ the step function. The KMS property is

$$M_\beta^{\alpha\alpha'}(p;e,e') = e^{\beta p^0}M_\beta^{\alpha'\alpha}(-p;e',e). \tag{16}$$

The existence of a KMS state is the prerequisite for the thermalization of a system. In his discussion of the possible physical significance of his zero mass infinite spin representations in [8] Wigner expressed concern about the infinite degeneracy of each energy level in the one-particle space, that apparently would lead to a divergence of the partition function in a box. It is not clear, however, if such a treatment is legitimate for objects with a semi-infinite string localization. This question merits a further study, including a comparison with the results of [22] on the thermodynamic properties of conventional quantum fields.

An important open problem in this context is the existence of local observables in the sense of [23], i.e., operators that are localized in bounded domains of Minkowski space and relatively local for the fields. From the modular duality results of [7] it follows that such operators must be contained in the intersection of the operator algebras generated by string field operators localized in wedge domains containing the bounded localization domain, so the question is whether the intersections of the wedge algebras contain nontrivial local operators. A sufficent condition based on nuclearity properties of modular operators has very recently been given in [18] but it is restricted to space-time dimensions not larger than two and hence not applicable in the present case without modifications.

Our results suggest that although string-localized fields are admitted by the physical principles, they are outside the realm of Lagrangean quantization and hence call for new concepts and methods which are more intrinsically rooted in local quantum physics [31]. As a historical remark we point out that already in 1929 Pascual Jordan made a plea for an instrinsic formulation of QFT without *"klassisch-korrespondenzmässige Krücken"* (quasiclassical crutches) [24]. The concept of modular localization, that inspired the present construction of string localized fields, can be regarded as a modern realization of this vision of Jordan. One of its achievements is the successful derivation, from first principles, of the recipes of the bootstrap-formfactor programs for the rich class of $d = 1+1$ factorising models [17, 18]. What has been missing up to now is an example demonstrating beyond doubt that this trans-Lagrangean point of view is also relevant in four space-time dimensions. Our string localized fields provide such examples. Furthermore, current work [20] indicates that our construction has the potential for further applications. Namely, along the same lines string localized fields can be constructed also for massive particle types, opening the possibility for more general kinds of interactions than for the usual point-like fields. Note in this context that the results of [7, 11] support the viewpoint that

localization of quantum fields in space–like cones (the idealizations of which are our strings) is a natural concept, yet there is so far a lack of rigorous model realizations [32]. Now if there is an interacting quantum field with such localization, then the corresponding in- and out- fields, in the sense of LSZ, must be string–localized as well. Hence, our fields (in contrast to the usual free fields) may serve as the in- and out-fields of such a model. We shall return to this issue elsewhere [20].

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Appendix: Modular localization

For convenience of the reader we summarize here some basic definitions and facts about modular localization, referring to [7, 16, 19] for details. In a nutshell the idea is that there is a natural concept of localization of state vectors in space and time that is defined for certain representations of the Poincaré group. This concept has its roots in the CPT theorem and an important paper [28] of Bisognano and Wichmann. It is distinct from Newton-Wigner localization and not associated with any position operators (that are known to be problematic in relativistic quantum mechanics). One first definies localization in space like wedges and then carries the definition over to more general domains by forming intersections.

Let W be a space-like wedge, i.e., a Poincaré transform of the standard wedge $W_3 \equiv \{x = (x^0, \dots, x^3) \in \mathbb{R}^4 : |x^0| < x^3\}$. To W belongs a one-parameter family $\Lambda_W(t)$ of Lorentz boosts that leave W invariant (t is the rapidity parameter), and a reflection j_W about the edge of the wedge that maps W into the opposite wedge W'. (The dependence of these transformations on W was suppressed in Eq. (14).)

Let U be a representation of the proper Poincaré group $\mathcal{P}_+$ on a Hilbert space $\mathcal{H}$. It is assumed that U is unitary on the orthocronous group $\mathcal{P}_+^\uparrow$ but antiunitary for the reflections j_W. Moreover, the energy spectrum is assumed to be nonnegative.

For a given wedge W, the "modular operator" Δ_W is defined as the unique positive operator satisfying $\Delta_W^{it} = U(\Lambda_W(-2\pi t))$ for all real t. It is an unbounded operator (exept

in trivial cases) and hence can not be defined on the whole of $\mathcal{H}$. The same applies to $\Delta_W^{1/2}$ which has a natural domain of definition, $D(W) \subset \mathcal{H}$. Concretely, $D(W)$ consists of state vectors $\psi \in \mathcal{H}$ such that $U(\Lambda_W(-2\pi t))\psi$ can be analytically continued to the strip $0 \leq \operatorname{Im} t \leq \pi$.

Let J_W to be the anti–unitary operator representing j_W. The operator $S_W \equiv J_W\, \Delta_W^{1/2}$ ("Tomita conjugation") is defined on $D(W)$ and satisfies $S_W^2 \subset$id. State vectors left invariant under S_W, i.e., belonging to the real subspace

$$\mathcal{K}(W) \equiv \{\phi \in D(W) : S_W \phi = \phi\} \tag{17}$$

are said to be *localized* in the wedge W in the modular sense. The space $\mathcal{K}(W)$ is a real Hilbert space with the real scalar product $\operatorname{Re}(\psi, \phi)$. Moreover, it satisfies $\mathcal{K}(W) \cap \mathrm{i}\mathcal{K}(W) = \{0\}$, and $\mathcal{K}(W) + \mathrm{i}\mathcal{K}(W)$ is dense in $\mathcal{H}$.

The localization attribute is justified by the fact that the symplectic complement

$$\mathcal{K}(W)' \equiv \{\psi :\ \operatorname{Im}(\psi, \phi) = 0 \,\text{for all}\, \phi \in \mathcal{K}(W)\} \tag{18}$$

is equal to $\mathcal{K}(W')$, i.e., the space of state vectors localized in the causal complement of W. Second quantization allows one to define field operators $\Phi(\psi)$ on the Fock space over $\mathcal{H}$ such that $[\Phi(\psi), \Phi(\phi)] = \mathrm{i} \operatorname{Im}(\psi, \phi)$, and by Eq. (18) $\Phi(\psi)$ and $\Phi(\phi)$ commute if ψ and ϕ are localized in causally separated wedges.

For more general domains $G \subset \mathbb{R}^4$ one defines the corrsponding spaces $\mathcal{K}(G)$ of localized vectors as the intersections of the spaces $\mathcal{K}(W)$ for all wedges W containing G. But while $\mathcal{K}(W)$ is always large in the sense that $\mathcal{K}(W) + \mathrm{i}\mathcal{K}(W)$ is dense in $\mathcal{H}$, this is in general not so for $\mathcal{K}(G)$ which may consist only of the zero vector. It is a highly nontrivial result of [7] that $\mathcal{K}(G) + \mathrm{i}\mathcal{K}(G)$ is still dense if G is a spacelike cone, i.e., a set of the form $x + \{\lambda y :\ \lambda > 0, y \in B\}$ where $x \in \mathbb{R}^4$ and B is an (arbitrarily small) open set not containig the origin.

The string localized fields (4) realize these ideas in a concrete setting. The discussion following Eq. (14) confirms implicitly that the fields (4) generate states that are localized in the modular sense.

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* Electronic address: mund@fma.if.usp.br

† Electronic address: schroer@physik.fu-berlin.de

‡ Electronic address: yngvason@thor.thp.univie.ac.at

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[29] The distributional character of the fields requires in fact strict separation in the sense that that some open neighborhoods of the strings are space-like separated.

[30] A closely related use of representations of the homogeneous Lorentz group is made in [5]. The essential difference is that string-localization, which is our main concern, is not visible in this earlier construction.

[31] Looking with the present hindsight (of quantum localizability in the representation theoretical setting) at the early history of quantized fields which eventually culminated in renormalized perturbation theory, it appears as an instance of undeserved luck that the point-like quantum localizability in Wigner's representation theoretical approach for particles with finite (half)integer spin/helicity made such a perfect match with the locality inherent in the classical formalism of local tensor/spinor fields.

[32] apart from non–Lorentz covariant infra–vacua models as in [25] and lattice models as in [26, 27].